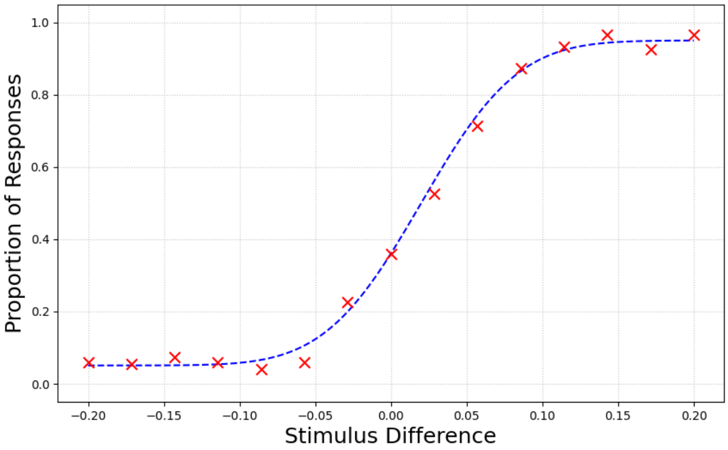
A costly-information foundation for psychometric curve

Jake Zhang

May 30, 2026

Motivation

- Perception comes before action
  - firm fundamentals $\rightarrow$ investment, candidate quality $\rightarrow$ recruitment, symptom $\rightarrow$ prescription, etc
- Psychophysics studies how we perceive
  - use perceptual tasks, e.g., brightness, pitch, weight, motion
  - summarize data by psychometric function
- Sigmoidal curves are widely observed
  - Fechner (1860)' law: subjective sensation is proportional to the logarithm of the stimulus intensity
  - Recent explanation: Signal detection theory, Drift diffusion model, and Efficient coding
- Micro foundation: optimal trade-off between benefit and cost of information processing



1.0
0.8
0.6
0.4
0.2
0.0
Proportion of Responses
−0.20
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15
0.20
Stimulus Difference

Rational Inattention Framework

Decision maker

- faces state-contingent payoffs and uncertainty about the state
  - model stimulus strength as the state
- chooses a signal structure with a certain cost
  - measured by Fisher information cost HW2021
- takes action optimally according to the received signal
  - equivalently, state-dependent choice rules/response curve
- chooses optimally to maximize the gain minus cost

Main result: under uniform prior and increasing relative payoff, any (interior) optimal response curve is sigmoidal.

Fisher information cost

- Fisher information captures local distinguishability,

$$\mathcal{I}_F(\{\sigma_\theta\}) := E_s[(\frac{\partial}{\partial\theta}\log\sigma(s|\theta))^2]$$

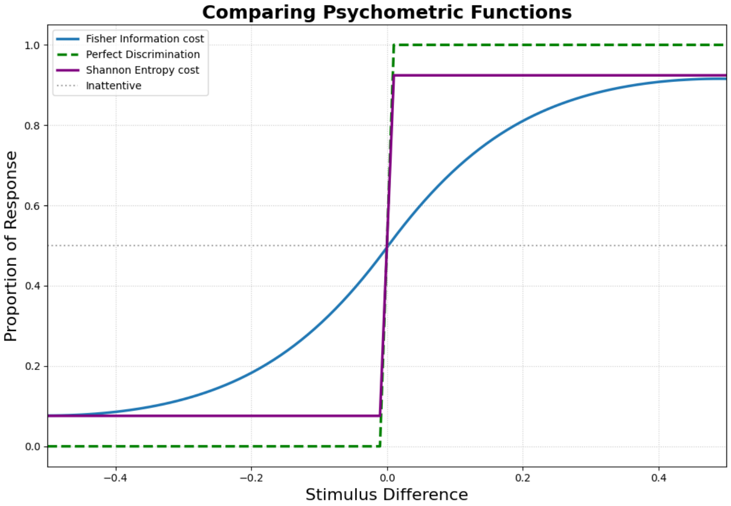
- Fisher information cost measures overall responsiveness

$$C_{FI}(\sigma) := E_{\pi(\theta)}[\mathcal{I}_F(\{\sigma(\theta)\})] = \int_\Theta \pi(\theta)\sum_{s\in S}\frac{(\frac{\partial}{\partial\theta}\sigma(s|\theta))^2}{\sigma(s|\theta)}d\theta.$$

- Penalizes sharp distinctions between nearby states, especially those near 0/1

Why Fisher information cost?

- Shannon model fails to capture perceptual distance and overestimates DM's response to incentives (Dean and Neligh, 2023)
- Fisher information cost can be considered as continuous extension of *Neighborhood-based* cost (Hébert and Woodford, 2021), *Log-Likelihood* cost (Pomatto et al., 2023), *Total Information* cost (Bloedel and Zhong, 2024), *f-Information* cost (Bloedel et al., 2025)
  - Fisher information is the 2nd order gradient of f-divergence


Comparing Psychometric Functions
Fisher Information cost
Perfect Discrimination
Shannon Entropy cost
Inattentive
1.0
0.8
0.6
0.4
0.2
0.0
−0.4
−0.2
0.0
0.2
0.4
Proportion of Response
Stimulus Difference

A binary choice problem (2AFC)

- State space: Θ, (full-support) Prior: $\pi \in \Delta(\Theta)$
- Action Set: $A = \{L, R\}$
- State-contingent payoff: $u(t, L) \equiv 0$, denote $u(t, R) := u(t)$
- Optimal signal is binary: $p(t) := \sigma(R|t)$ for the probability of getting a recommendation of taking R under state t
- Expected gain :

$$\int_{\Theta} \pi(t)u(t)p(t)dt$$

- Cost of information:

$$C_{FI}(p, \pi) = \int_{\Theta} \pi(t) \frac{(p'(t))^2}{p(t)(1-p(t))} dt,$$

we treat

$$\frac{(p'(t))^2}{p(t)(1-p(t))} = \begin{cases} \frac{(p'(t))^2}{p(t)(1-p(t))}, & \text{if } p(t) \in (0,1) \\ 0, & \text{if } p'(t) = 0 \text{ and } p(t) = 0 \text{ or } 1 \,. \\ +\infty & \text{other cases} \end{cases}$$

The Optimization Problem

DM chooses a state-contigent response rule $p(t)$:

$$\underset{p \in AC(\Theta,[0,1])}{\text{maximize}} \int_{\Theta} \pi(t)u(t)p(t)dt - \lambda C_{FI}(p,\pi)$$

$$\Leftrightarrow \underset{p \in AC(\Theta,[0,1])}{\text{minimize}} \int_{\Theta} \lambda\pi(t)\frac{(p'(t))^2}{p(t)(1-p(t))} - \pi(t)u(t)p(t)dt\} \qquad (P)$$

where λ is a scaling parameter.

Simplification

- normalize the scaling parameter $\lambda = 1$
- normalize the state space $\Theta = [0, 1]$
- assume the prior π is uniform

Reparametrization

We consider the following change of variable:

$$p(t) = \frac{1+\sin(w(t))}{2}, p(t) \in [0,1] \Leftrightarrow w(t) = \arcsin(2p(t)-1), w(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Our problem becomes [1]:

$$\underset{w \in AC(\Theta,[-\frac{\pi}{2},\frac{\pi}{2}])}{\text{minimize}} \int_\Theta \lambda(w'(t))^2 - u(t)\frac{1+\sin(w(t))}{2}\, dt. \qquad (W)$$

We will apply tools from the **Calculus of Variations** to solve the above problem: existence, uniqueness, regularity and necessary conditions, .

[1] To see this note that, $p'(t) = \frac{1}{2}\cos(w(t))w'(t)$ and $p(t)(1-p(t)) = \frac{1}{4}(1-\sin^2(w(t)))$

Existence

Theorem (Existence)

Suppose u is piecewise continuous, then Problem (P) admits a solution in $Lip([0,1],[0,1])$. Furthermore, if u is also single-crossing, then the optimal solution to Problem (P) belongs to one of the following two kinds:

- Pure boundary solution: $p(t) = 0$ for all $t \in [0,1]$ or $p(t) = 1$ for all $t \in [0,1]$
- Strict interior solution: $0 < p(t) < 1$ for all $t \in [0,1]$

Uniqueness

Theorem (Uniqueness)

If the problem admits a non-constant interior optimal solution, then it is the unique solution.

Because the integrand $\frac{p'(t)^2}{p(t)(1-p(t))} - u(t)p(t)$ is convex in p. Two different continuous functions that differ in a neighborhood of positive measure imply that their convex combination attains a lower value.

Regularity

Theorem (Regularity)

Let p_* be a solution to Problem (P). If u is piecewise continuous, then p_* is twice differentiable almost everywhere. Furthermore, if $u \in C^k$ for some $k \in \mathbb{N}$, then $p_* \in C^{k+2}$.

As a result, the response curve exhibits no kinks or spikes.

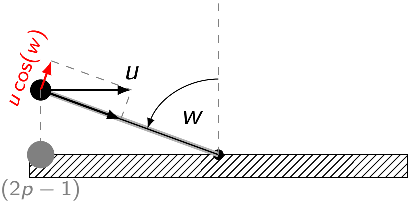
Characterizing the optimal (interior) solution

Theorem (Necessity)

Suppose u is piecewise continuous on $\bigcup_{i=0}^{N}(t_i, t_{i+1})$. Let w_* be a solution to Problem (W), then w_* satisfies the **integral Euler equation** together with **natural boundary conditions** and is continuously differentiable at each discontinuity of u:

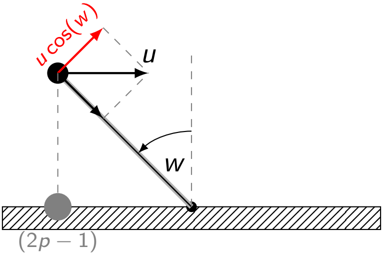
$$\begin{cases} w_*'(t) = \int_0^t -\frac{1}{4} u(s) \cos(w_*(s)) ds \quad \text{a.e. on } [0,1] \\ w_*'(0) = w_*'(1) = 0 \end{cases} \tag{NC}$$

Connection to a pendulum system



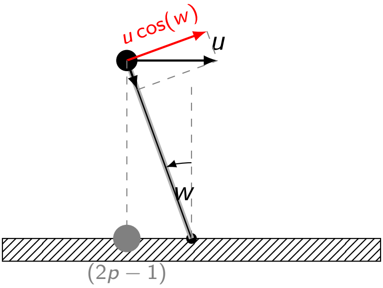
- $w(t)$: angle between rod and vertical axis. Angles to the left of the vertical axis are negative
- $u(t)$: magnitude of time-varying horizontal force. A force directed to the right is assigned a negative sign.
- Task: choose initial angle w_0 so that angular velocity is zero at $t = 0$ and $t = 1$
  $\Leftarrow w''(t) = -\frac{1}{4}u(t)\cos(w(t)) \quad w'(0) = w'(1) = 0$

Connection to a pendulum system



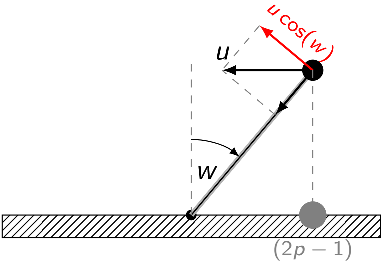
- $w(t)$: angle between rod and vertical axis. Angles to the left of the vertical axis are negative
- $u(t)$: magnitude of time-varying horizontal force. A force directed to the right is assigned a negative sign.
- Task: choose initial angle w_0 so that angular velocity is zero at $t = 0$ and $t = 1$
  $\Leftarrow w''(t) = -\frac{1}{4}u(t)\cos(w(t)) \quad w'(0) = w'(1) = 0$

Connection to a pendulum system



- $w(t)$: angle between rod and vertical axis. Angles to the left of the vertical axis are negative
- $u(t)$: magnitude of time-varying horizontal force. A force directed to the right is assigned a negative sign.
- Task: choose initial angle w_0 so that angular velocity is zero at $t = 0$ and $t = 1$
  $\Leftarrow w''(t) = -\frac{1}{4}u(t)\cos(w(t)) \quad w'(0) = w'(1) = 0$

Connection to a pendulum system



- $w(t)$: angle between rod and vertical axis. Angles to the left of the vertical axis are negative
- $u(t)$: magnitude of time-varying horizontal force. A force directed to the right is assigned a negative sign.
- Task: choose initial angle w_0 so that angular velocity is zero at $t = 0$ and $t = 1$
  $\Leftarrow w''(t) = -\frac{1}{4}u(t)\cos(w(t)) \quad w'(0) = w'(1) = 0$

Predicting Psychometric curve

Proposition

If payoff u is single–crossing from below, then the optimal p_* is increasing.
Furthermore, suppose

- the prior is uniform
- the payoff u is increasing
- interior solution exists

then, the optimal solution p^* exhibits sigmoidal shape.

- The object first accelerates, then decelerates, and is always moving clockwise
- Its shadow accelerates at start and decelerates at the final

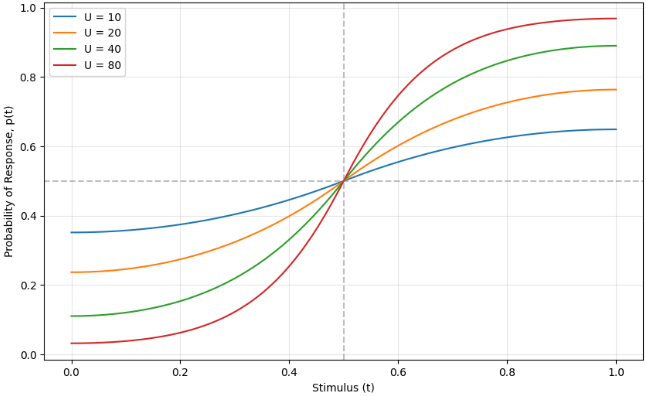
Comparative statics

Proposition

Suppose the prior is uniform, the payoff is anti-symmetric and single-crossing,
If the cost is scaled down $0 < \lambda < 1$:, then:

- the response becomes more sensitive around the threshold $p_*'(\frac{1}{2}) \uparrow$
- the curve moves closer to correct responses $|p_*(t) - \frac{1}{2}| \uparrow$

- Stronger incentives implies stronger force, the object needs to start further away from the vertical line
- No overtake


U = 10
U = 20
U = 40
U = 80
1.0
0.8
0.6
0.4
0.2
0.0
Probability of Response, p(t)
0.0
0.2
0.4
0.6
0.8
1.0
Stimulus (t)

Takeaway

- Fisher information cost provides a new explnation to sigmoidal response curve in perception-like economic decisions
- Connection between the optimal condition and a simple pendulumn system

  $\rightarrow$ offer insight into the shape of the solution and comparative statics

Questions?

Thank you for your attention!

Any questions or comments?

References I

Bloedel, A., Denti, T., and Pomatto, L. (2025). Modeling information acquisition via f-divergence and duality. *arXiv preprint arXiv:2510.03482*.

Bloedel, A. W. and Zhong, W. (2024). The cost of optimally-acquired information. *Working paper*.

Dean, M. and Neligh, N. (2023). Experimental tests of rational inattention. *Journal of Political Economy*, 131(12):3415–3461.

Fechner, G. T. (1860). *Elemente der psychophysik*, volume 2. Breitkopf u. Härtel.

Hébert, B. and Woodford, M. (2021). Neighborhood-based information costs. *American Economic Review*, 111(10):3225–3255.

Pomatto, L., Strack, P., and Tamuz, O. (2023). The cost of information: The case of constant marginal costs. *American Economic Review*, 113(5):1360–1393.

Change-of-Variable trick

It is without loss to work on the re-parametrized problem:

Lemma

The transformation $\Phi(p) := \arcsin(2p-1)$ *and* $\Psi(w) := \frac{1+\sin(w)}{2}$ *defines a bijection between the two classes of admissible solutions:*

$$p = \frac{1+\sin(w)}{2} \in \mathcal{D}^P \quad \Longleftrightarrow \quad w = \arcsin(2p-1) \in \mathcal{D}^W.$$

where

$$\mathcal{D}^P := \{p \in AC(\Theta, [0,1]) : \int_\Theta \frac{(p'(\theta))^2}{p(\theta)(1-p(\theta))} d\theta < +\infty\}$$

$$\mathcal{D}^W := \{w \in AC(\Theta, [-\frac{\pi}{2}, \frac{\pi}{2}]) : \int_\Theta (w'(\theta))^2 d\theta < +\infty\}$$

Moreover, the value of the objective function is preserved under such a reparametrization, that is $\int_\Theta u(t)p(t) - \frac{p'(t)^2}{p(t)(1-p(t))} dt = \int_\Theta u(t)\frac{1+\sin(w(t))}{2} - (w'(t))^2 dt$ *if* $p = \frac{1+\sin(w)}{2}$ *and both are admissible.*

Regularity

Theorem (Regularity)

Let p_* be a solution to Problem (P). If u is piecewise continuous, then p_* is twice differentiable almost everywhere. Furthermore, if $u \in C^k$ for some $k \in \mathbb{N}$, then $p_* \in C^{k+2}$.

As a result, the response curve exhibits no kinks or spikes.

Example

Consider the payoff $u = \begin{cases} -60 \in [0, 0.35] \\ 30 \in (0.35, 1] \end{cases}$, as can be seen from the numerical solution in Figure 1, $p''_*(t)$ has discontinuity at the point where u is discontinuous.

Regularity

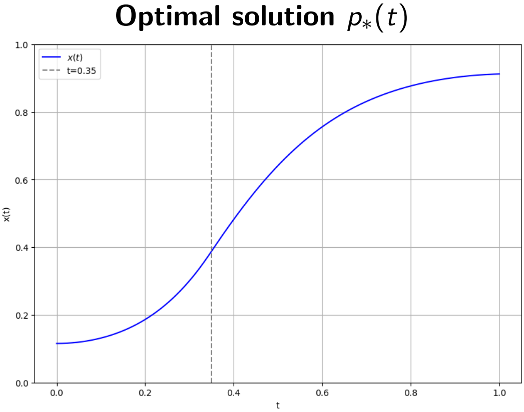


Figure: Example where $u \in$ PWC implies p_* twice differentiable almost everywhere

Regularity

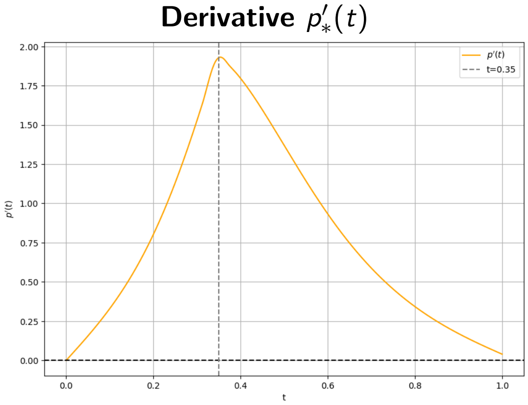


Figure: Example where $u \in$ PWC implies p_* twice differentiable almost everywhere

Regularity

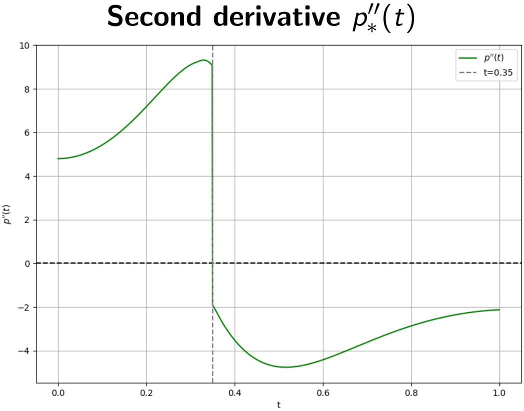


Figure: Example where $u \in \mathrm{PWC}$ implies p_* twice differentiable almost everywhere

Predicting Psychometric curve

Proposition

Suppose the prior is uniform over Θ, the payoff is antisymmetric and single-crossing, i.e. $u(t) = -u(1-t)$ on $t \in [0,1]$ and $u(t) < 0$ on $[0,1/2)$, then the optimal solution is interior. Let p_* be the interior maximizer of Problem (P). Then:

$$p_*''(t) > 0 \text{ on } [0,1/2), \text{ and } p_*''(t) < 0 \text{ on } (1/2,1].$$

2AFC

Example (2AFC)

Suppose the subject is rewarded for correct answers, then
$u(t) = \begin{cases} -U, & t \in [0, 0.5), \\ U, & t \in (0.5, 1], \end{cases}$,
We have analytical solutions for the ODE of the simple pendulum:

$$w''(t) + \frac{U}{4}\cos(w(t)) = 0$$

We just need to choose $w(0)$ and $w(1)$ to match the two curves:

$$w(0.5^-) = w(0.5^+) \quad w'(0.5^-) = w'(0.5^+)$$